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On the general structure of nonlinear evolution equations and their Bäcklund transformations connected with the matrix non-stationary Schrödinger spectral problem

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Abstract. The general form of nonlinear evolution equations in $1+2$ dimensions integrable by the matrix non-stationary Schrödinger spectral problem is found. The infinite-dimensional group of Bäcklund transformations for these equations is constructed and the nonlinear superposition principle is obtained.

1. Introduction

One of the main problems of the inverse scattering transform (IST) method is that of the description of the equations integrable by this method (see e.g. Zakharov *et al* 1980, Bullough and Caudrey 1980). In Ablowitz *et al* (1974, to be referred to as AKNS) the very simple and convenient description of the class of partial differential equations integrable by the second-order problem (*) $\partial\psi/\partial x = \lambda A\psi + p(x, t)\psi$ has been given. Then this approach (the AKNS approach) was generalised to the problem (*) of arbitrary order (Miodek 1978, Newell 1979, Kulish 1980, Konopelchenko 1980a, b, c, 1981a, b) and to some other one-dimensional spectral problems, Konopelchenko 1981c, d, Gerdjikov *et al* 1980). The infinite-dimensional groups of Bäcklund transformations for these classes of integrable equations have been also found (Calogero and Degasperis 1976, 1977, Gerdjikov *et al* 1980, Konopelchenko 1980a, b, c, 1981a, b, c, d).

The generalisation of the AKNS approach to the two-dimensional arbitrary-order spectral problem $\partial\psi/\partial x + A\partial\psi/\partial y + p(x, y, t)\psi = 0$ where A is a diagonalisable matrix has recently been done by Konopelchenko (1981d). In Konopelchenko (1981d) the general form of the integrable equations in $1+2$ dimensions (one time and two spatial dimensions) and their Bäcklund transformations were obtained.

In the present paper we consider the two-dimensional matrix spectral problem

$$\alpha\frac{\partial\psi}{\partial y}+\frac{\partial^2\psi}{\partial x^2}+U(x, y, t)\psi = 0 \tag{1.1}$$

where α is an arbitrary constant, the potential $U(x, y, t)$ is an $N\times N$ matrix and $U(x, y, t)\to 0$ ($\sqrt{x^2+y^2}\to\infty$). The order, N, of the matrix $U(x, y, t)$ is arbitrary. The spectral problem (1.1), i.e. the non-stationary Schrödinger spectral problem is well known. In the scalar case ($N=1$) it was used for integration of the Kadomtsev–

Petviashvili equation (Zakharov and Shabat 1974, Dryuma 1974, Zakharov and Manakov 1979, Zakharov 1980, Manakov 1981).

In this paper we find the general form of the nonlinear evolution equations in $1+2$ dimensions integrable by (1.1). We construct the infinite-dimensional group of Bäcklund transformations for these equations. We also obtain the nonlinear superposition formula for the simplest Bäcklund transformation.

The paper is organised as follows. In § 2 we introduce some special solutions of the linear problem equivalent to (1.1), the scattering matrix, and obtain several important relations. In § 3 we calculate the recursion operators $\hat{\Lambda}^{+}_{(n)}$ and $\check{\Lambda}^{+}_{(n)}$ which play a fundamental role in our constructions. The general form of the integrable equations and Bäcklund transformations is found in § 4. In § 5 we consider the simplest Bäcklund transformation and obtain the nonlinear superposition formulae.

2. Some preliminary relations

First of all let us note that the non-stationary Schrödinger problem (1.1) is equivalent to the $2N$-order linear problem

$$\frac{\partial \hat{F}}{\partial x}+\begin{pmatrix} 0 & \alpha I_N \\ 0 & 0 \end{pmatrix}\frac{\partial \hat{F}}{\partial y}+\begin{pmatrix} 0 & U(x,y,t) \\ -I_N & 0 \end{pmatrix}\hat{F}=0 \tag{2.1}$$

where I_N is an identical $N\times N$ matrix and 0 denotes an $N\times N$ matrix with zero elements.

Together with the problem (2.1) one must also consider the adjoint problem

$$\frac{\partial \check{F}}{\partial x}+\frac{\partial \check{F}}{\partial y}\begin{pmatrix} 0 & \alpha I_N \\ 0 & 0 \end{pmatrix}+\check{F}\begin{pmatrix} 0 & -U(x,y,t) \\ I_N & 0 \end{pmatrix}=0. \tag{2.2}$$

It is more convenient, for our purposes, to consider the problems (2.1) and (2.2) than the initial problem (1.1) and its adjoint problem $-\alpha\,\partial\check{\psi}/\partial y+\partial^2\check{\psi}/\partial x^2+\check{\psi}U(x,y,t)=0$.

Let us introduce, following from Zakharov (1980), Bullough and Caudrey (1980) and Konopelchenko (1981d), the matrix solutions $\hat{F}^{+}_{\lambda}(x,y,t)$ and $\hat{F}^{-}_{\lambda}(x,y,t)$ of the problem (2.1) given by their asymptotic behaviour

$$\begin{aligned} \hat{F}^{+}_{\lambda}(x,y,t) &\xrightarrow[x\to+\infty]{} (2\pi \mathrm{i})^{-1/2}\mathscr{D}(\lambda)\exp(\lambda^2 y-\mathrm{i}\sqrt{\alpha}\lambda\sigma x) \\ \hat{F}^{-}_{\lambda}(x,y,t) &\xrightarrow[x\to-\infty]{} (2\pi \mathrm{i})^{-1/2}\mathscr{D}(\lambda)\exp(\lambda^2 y-\mathrm{i}\sqrt{\alpha}\lambda\sigma x) \end{aligned} \tag{2.3}$$

where λ is a complex number,

$$\sigma=\begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix} \qquad \text{and} \qquad \mathscr{D}(\lambda)=\begin{pmatrix} I_N, & \mathrm{i}\sqrt{\alpha}I_N\lambda \\ (\mathrm{i}/\sqrt{\alpha}\lambda)I_N, & I_N \end{pmatrix}.$$

Note that

$$\mathscr{D}^{-1}\begin{pmatrix} 0 & -\alpha\lambda^2 I_N \\ I_N & 0 \end{pmatrix}\mathscr{D}=-\mathrm{i}\sqrt{\alpha}\lambda\sigma.$$

The scattering matrix $\hat{S}(\tilde{\lambda}, \lambda, t)$ is defined as follows (Zakharov 1980, Bullough and Caudrey 1980, Konopelchenko 1981d)

$$\hat{F}^+_\lambda(x, y, t) = \int_{-\infty}^{+\infty} d\tilde{\lambda}\, \hat{F}^-_{\tilde{\lambda}}(x, y, t)\hat{S}(\tilde{\lambda}, \lambda, t). \tag{2.4}$$

Correspondingly for the adjoint problem (2.2) we introduce the matrix solutions $\check{F}^+_\lambda(x, y, t)$ and $\check{F}^-_\lambda(x, y, t)$

$$\begin{aligned} \check{F}^+_\lambda(x, y, t) &\xrightarrow[x\to+\infty]{} (2\pi i)^{-1/2} \exp(-\lambda^2 y + i\sqrt{\alpha}\lambda\sigma x)\mathscr{D}^{-1}(\lambda) \\ \check{F}^-_\lambda(x, y, t) &\xrightarrow[x\to-\infty]{} (2\pi i)^{-1/2} \exp(-\lambda^2 y + i\sqrt{\alpha}\lambda\sigma x)\mathscr{D}^{-1}(\lambda) \end{aligned} \tag{2.5}$$

and the scattering matrix $\check{S}(\tilde{\lambda}, \lambda, t)$:

$$\check{F}^+_\lambda(x, y, t) = \int_{-\infty}^{+\infty} d\tilde{\lambda}\, \check{S}(\lambda, \tilde{\lambda}, t)\check{F}^-_{\tilde{\lambda}}(x, y, t). \tag{2.6}$$

One can show with the use of (2.1)–(2.6) that the following relations hold

$$\begin{aligned} &\int_{-\infty}^{+\infty} dy\, \check{F}^\pm_{\tilde{\lambda}}(x, y, t)\hat{F}^\pm_\lambda(x, y, t) = \delta(\tilde{\lambda} - \lambda) \\ &\int_{-\infty}^{+\infty} d(\lambda^2)\, \hat{F}^\pm_\lambda(x, y, t)\check{F}^\pm_\lambda(x, y', t) = \delta(y' - y) \\ &\int_{-\infty}^{+\infty} d\mu\, \check{S}(\tilde{\lambda}, \mu, t)\hat{S}(\mu, \lambda, t) = \delta(\tilde{\lambda} - \lambda) \end{aligned} \tag{2.7}$$

where $\delta(\lambda)$ is the Dirac delta function.

We assume that the potential $U(x, y, t)$ decreases as $\sqrt{x^2+y^2} \to \infty$ so fast that all the integrals which will appear in our calculations exist and that $\int_{-\infty}^{+\infty} dy\, \partial(\ldots)/\partial y = 0$.

Now, let

$$P = \begin{pmatrix} 0 & U \\ -I_N & 0 \end{pmatrix} \qquad \text{and} \qquad P' = \begin{pmatrix} 0 & U' \\ -I_N & 0 \end{pmatrix}$$

be two different potentials and $\hat{F}^+$, $\check{F}^+$, $\hat{F}^{+\prime}$, $\hat{S}$, $\hat{S}'$ be corresponding solutions and scattering matrices of the problems (2.1) and (2.2). One can prove (analogously to Konopelchenko (1981d)) the following important relation

$$\begin{aligned} \hat{S}'(\tilde{\lambda}, \lambda, t) - \hat{S}(\tilde{\lambda}, \lambda, t) = -\int_{-\infty}^{+\infty} d\mu\, \hat{S}(\tilde{\lambda}, \mu, t) \int_{-\infty}^{+\infty} dx\, dy\, \check{F}^+_\mu(x, y, t) \\ \times (P'(x, y, t) - P(x, y, t))\hat{F}^{+\prime}_\lambda(x, y, t). \end{aligned} \tag{2.8}$$

The mapping $U(x, y, t) \to \hat{S}(\tilde{\lambda}, \lambda, t)$ given by the spectral problem (2.1) and formula (2.8) establishes a correspondence between the transformations $T_u : U \to U'$ on the manifold of potentials $\{U(x, y, t),\ U(x, y, t) \to 0,\ \sqrt{x^2+y^2} \to \infty\}$ and the transformations $T_s : \hat{S} \to \hat{S}'$ on the manifold of scattering matrices $\{\hat{S}(\tilde{\lambda}, \lambda, t)\}$.

Let us consider only transformations T such that

$$\hat{S}(\tilde{\lambda}, \lambda, t) \xrightarrow{T_s} \hat{S}'(\tilde{\lambda}, \lambda, t) = B^{-1}(\tilde{\lambda}, t)\hat{S}(\tilde{\lambda}, \lambda, t)C(\lambda, t) \tag{2.9}$$

where

$$B(\lambda, t) = \begin{pmatrix} B_1(\lambda^2, t) - i\sqrt{\alpha}\lambda B_2(\lambda^2, t) & 0 \\ 0 & B_1(\lambda^2, t) + i\sqrt{\alpha}\lambda B_2(\lambda^2, t) \end{pmatrix}$$

$$C(\lambda, t) = \begin{pmatrix} C_1(\lambda^2, t) - i\sqrt{\alpha}\lambda C_2(\lambda^2, t) & 0 \\ 0 & C_1(\lambda^2, t) + i\sqrt{\alpha}\lambda C_2(\lambda^2, t) \end{pmatrix}$$

and $B_1(\lambda^2, t)$, $B_2(\lambda^2, t)$, $C_1(\lambda^2, t)$, $C_2(\lambda^2, t)$ are arbitrary matrices of the order N. These 'restricted' transformations of the form (2.9) are, as we shall see, wide enough.

It is not difficult to show that the following identity holds

$$\begin{aligned} -\int_{-\infty}^{+\infty} d\mu\, \check{S}(\tilde{\lambda}, \mu, t)(1 - B(\mu, t))\hat{S}'(\mu, \lambda, t) + (1 - B(\lambda, t))\delta(\tilde{\lambda} - \lambda) \\ = \int_{-\infty}^{+\infty} dx\, dy \frac{\partial}{\partial x}\left[\check{F}_{\tilde{\lambda}}^{+}(x, y, t)\left(1 - \tilde{B}\left(\frac{\partial}{\partial y}, t\right)\right)\hat{F}_{\lambda}^{+\prime}(x, y, t)\right] \\ = -\int_{-\infty}^{+\infty} dx\, dy\, \check{F}_{\tilde{\lambda}}^{+}(x, y, t)\left[\tilde{P}(x, y, t)\left(1 - \tilde{B}\left(\frac{\partial}{\partial y}, t\right)\right)\hat{F}_{\lambda}^{+\prime}(x, y, t) \right. \\ \left. - \left(1 - \tilde{B}\left(\frac{\partial}{\partial y}, t\right)\right)\tilde{P}'(x, y, t)\hat{F}_{\lambda}^{+\prime}(x, y, t)\right] \end{aligned} \tag{2.10}$$

where

$$\tilde{P} = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}$$

and

$$\tilde{B}\left(\frac{\partial}{\partial y}, t\right) = \begin{pmatrix} B_1(\partial/\partial y, t), & -\alpha(\partial/\partial y)B_2(\partial/\partial y, t) \\ B_2(\partial/\partial y, t), & B_1(\partial/\partial y, t) \end{pmatrix}.$$

Combining the relations (2.8), (2.9) and taking into account the identity (2.10) we find

$$\begin{aligned} \int_{-\infty}^{+\infty} dx\, dy \left[\check{F}_{\tilde{\lambda}}^{+}(x, y, t)\tilde{B}\left(\frac{\partial}{\partial y}, t\right)\tilde{P}'(x, y, t)\hat{F}_{\lambda}^{+\prime}(x, y, t) \right. \\ \left. - \tilde{P}(x, y, t)\tilde{B}\left(\frac{\partial}{\partial y}, t\right)\hat{F}_{\lambda}^{+\prime}(x, y, t)\right]_F = 0. \end{aligned} \tag{2.11}$$

Here and in what follows for an arbitrary $2N \times 2N$ matrix

$$\phi = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{pmatrix}$$

where ϕ_1, ϕ_2, ϕ_3, ϕ_4 are $N \times N$ matrices, the quantity ϕ_F means

$$\phi_F \overset{\text{def}}{=} \begin{pmatrix} 0 & \phi_2 \\ \phi_3 & 0 \end{pmatrix}.$$

The matrices $B_1(\partial/\partial y, t)$ and $B_2(\partial/\partial y, t)$ which are contained in formula (2.11) can be represented by the form $B_1(\partial/\partial y, t) = \sum_{\gamma=1}^{N^2} B_{1\gamma}(\partial/\partial y, t)H_\gamma$ and $B_2(\partial/\partial y, t) = \sum_{\gamma=1}^{N^2} B_{2\gamma}(\partial/\partial y, t)H_\gamma$ where matrices H_γ ($\gamma = 1, \ldots, N^2$) form a basis for the full linear matrix algebra gl(N, C) and $B_{1\gamma}(\partial/\partial y, t), B_{2\gamma}(\partial/\partial y, t)$ are some functions. In the present paper we shall consider only the functions $B_{1\gamma}(\partial/\partial y, t)$, $B_{2\gamma}(\partial/\partial y, t)$ entire on the first argument, i.e. $B_{1\gamma}(\partial/\partial y, t)=\Sigma_{n=0}^{\infty} b_{1\gamma(n)}(t)\partial^n/\partial y^n$, $B_{2\gamma}(\partial/\partial y, t)=\Sigma_{n=0}^{\infty} b_{2\gamma(n)}(t)\partial^n/\partial y^n$ where $b_{1\gamma(n)}(t)$ and $b_{2\gamma(n)}(t)$ are arbitrary functions. For such functions $B_{1\gamma}(\partial/\partial y, t)$ and $B_{2\gamma}(\partial/\partial y, t)$, the equality (2.11) can be rewritten as follows

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{d}y \sum_{\gamma=1}^{N^2} \sum_{n=0}^{\infty} \mathrm{Tr}\{U(x, y)H_\gamma b_{1\gamma(n)}(t)\hat{\phi}_{(n)}^{(F)} - H_\gamma U'(x, y)(-1)^n b_{1\gamma(n)}(t)\check{\phi}_{(n)}^{(F)}$$
$$+ U(x, y)H_\gamma b_{2\gamma(n)}(t)\hat{\chi}_{(n)}^{(F)} - H_\gamma U'(x, y) b_{2\gamma(n)}(t)(-1)^n \check{\chi}_{(n)}^{(F)}\} = 0 \tag{2.12}$$

where Tr denotes the usual matrix trace and

$$(\hat{\phi}_{(n)}^{(im)})_{kl} \overset{\text{def}}{=} \frac{\partial^n (\hat{F}_3^+)'_{ik}}{\partial y^n} (\check{F}_3^+)_{lm}, \qquad (\check{\phi}_{(n)}^{(im)})_{kl} \overset{\text{def}}{=} (\hat{F}_3^+)'_{ik} \frac{\partial^n (\check{F}_3^+)_{lm}}{\partial y^n}$$
$$(\hat{\chi}_{(n)}^{(im)})_{kl} \overset{\text{def}}{=} \frac{\partial^n (\hat{F}_1^+)'_{ik}}{\partial y^n} (\check{F}_3^+)_{lm}, \qquad (\check{\chi}_{(n)}^{(im)})_{kl} \overset{\text{def}}{=} (\hat{F}_3^+)'_{ik} \frac{\partial^n (\check{F}_4^+)_{lm}}{\partial y^n} \tag{2.13}$$
$$i, k, l, m = 1, \ldots, N; n = 0, 1, 2, \ldots$$

where we represent the $2N \times 2N$ matrices $\hat{F}^{+\prime}$ and $\check{F}^+$ in the block form

$$\hat{F}^{+\prime} = \begin{pmatrix} \hat{F}_1^{+\prime} & \hat{F}_2^{+\prime} \\ \hat{F}_3^{+\prime} & \hat{F}_4^{+\prime} \end{pmatrix} \qquad \check{F}^+ = \begin{pmatrix} \check{F}_1^+ & \check{F}_2^+ \\ \check{F}_3^+ & \check{F}_4^+ \end{pmatrix}.$$

3. Recursion operators

For further transformation of the equality (2.12) one must establish the relations between the quantities $\phi_{(n)}$ and $\chi_{(n)}$ with different n.

Let us introduce, in addition to $\hat{\phi}_{(n)}$ and $\hat{\chi}_{(n)}$, the quantities

$$(\hat{Z}_{1(n)}^{(im)})_{kl} \overset{\text{def}}{=} \frac{\partial^n (\hat{F}_3^+)'_{ik}}{\partial y^n} (\check{F}_4^+)_{lm}, \qquad (\hat{Z}_{2(n)}^{(im)})_{kl} \overset{\text{def}}{=} \frac{\partial^n (\hat{F}_1^+)'_{ik}}{\partial y^n} (\check{F}_4^+)_{lm}. \tag{3.1}$$

With the use of (2.1), (2.2) and definitions (2.13), (3.1) we obtain the following system of equations

$$\frac{\partial \hat{\phi}_{(n)}}{\partial x} = \hat{\chi}_{(n)} - \hat{Z}_{1(n)} \tag{3.2a}$$

$$\frac{\partial \hat{\chi}_{(n)}}{\partial x} = -\alpha \hat{\phi}_{(n+1)} - \hat{Z}_{2(n)} - \sum_{m=0}^{n} C_m^n U'_{(n-m)} \hat{\phi}_{(m)} \tag{3.2b}$$

$$\frac{\partial \hat{Z}_{1(n)}}{\partial x} = \alpha \hat{\phi}_{(n+1)} - \alpha \frac{\partial}{\partial y} \hat{\phi}_{(n)} + \hat{Z}_{2(n)} + \hat{\phi}_{(n)} U \tag{3.2c}$$

$$\frac{\partial \hat{Z}_{2(n)}}{\partial x} = -\alpha \hat{Z}_{1(n+1)} + \alpha \hat{\chi}_{(n+1)} - \alpha \frac{\partial}{\partial y} \hat{\chi}_{(n)} + \hat{\chi}_{(n)} U$$
$$- \sum_{m=0}^{n} C_m^n U'_{(n-m)} \hat{Z}_{1(m)} \qquad n = 0, 1, 2, \ldots \tag{3.2d}$$

where

$$C_m^n = \frac{n!}{m!(n-m)!} \qquad \text{and} \qquad U_{(k)} \overset{\text{def}}{=} \frac{\partial^k U(x, y, t)}{\partial y^k}.$$

Differentiating the equation (3.2a) over x and using (3.2b), (3.2c) one can express $\hat{Z}_{2(n)}$ through $\hat{\phi}_{(m)}$ $(m = 0, 1, \ldots, n+1)$. Substitution of the expressions obtained into equations (3.2b) and (3.2d) and the use of the equalities $\hat{\phi}^{(F)}_{(n)}\ (x = +\infty, y, t) = \hat{\chi}^{(F)}_{(n)}(+\infty, y, t) = \hat{Z}^{(F)}_{1(n)}(+\infty, y, t) = \hat{Z}^{(F)}_{2(n)}(+\infty, y, t) = 0$ give

$$\hat{\chi}^{(F)}_{(n)}(x, y, t) = \tfrac{1}{2}\partial^{-1}\left(\mathscr{L}^+\hat{\phi}^{(F)}_{(n)} - \sum_{m=0}^{n} C_m^n U'_{(n-m)}\hat{\phi}^{(F)}_{(m)}\right) \tag{3.3}$$

and

$$\begin{aligned}\hat{\phi}^{(F)}_{(n)} = \hat{\Lambda}_{(1)}\hat{\phi}^{(F)}_{(n-1)} - \frac{1}{4\alpha}\sum_{m=0}^{n-2} C_m^{n-1}\,\partial^{-1}\Big[\partial^{-1}\Delta_+(U'_{(n-1-m)}\hat{\phi}^{(F)}_{(m)}) - \partial^{-1}(U'_{(n-1-m)}\hat{\phi}^{(F)}_{(m)})U \\ + U'_{(n-1-m)}\,\partial^{-1}\Big(\Delta_+\hat{\phi}^{(F)}_{(m)} - \hat{\phi}^{(F)}_{(m)}\ U + \sum_{l=0}^{m} C_l^m U'_{(m-l)}\hat{\phi}^{(F)}_{(l)}\Big)\Big]\end{aligned} \tag{3.4}$$

where

$$(\partial^{-1}f)(x, y) \overset{\text{def}}{=} -\int_x^{\infty} \mathrm{d}x'\, f(x', y), \qquad \Delta_\pm = \pm\alpha\frac{\partial}{\partial y} + \frac{\partial^2}{\partial x^2}$$

$$\mathscr{L}\phi \overset{\text{def}}{=} \Delta_+\phi + U(x, y, t)\phi, \qquad \mathscr{L}^+\phi \overset{\text{def}}{=} \Delta_-\phi + \phi U(x, y, t)$$

and the operator $\hat{\Lambda}_{(1)}$ acts as follows

$$\hat{\Lambda}_{(1)}\phi = -\frac{1}{4\alpha}\partial^{-1}[\partial^{-1}(\Delta_-\mathscr{L}^+\phi + \Delta_+(U'\phi)) - \partial^{-1}(\mathscr{L}^+\phi - U'\phi)U - U'\partial^{-1}(\mathscr{L}'\phi - \phi U)]. \tag{3.5}$$

From (3.4) it follows that there exist operators $\hat{\Lambda}_{(n)}$ such that

$$\hat{\phi}^{(F)}_{(n)} = \hat{\Lambda}_{(n)}\phi^{(F)}_{(0)} \qquad n = 1, 2, 3, \ldots. \tag{3.6}$$

These operators $\hat{\Lambda}_{(n)}$ are determined from the following recursion relations

$$\begin{aligned}\hat{\Lambda}_{(n)}\phi = \hat{\Lambda}_{(1)}\hat{\Lambda}_{(n-1)}\phi \\ - \frac{1}{4\alpha}\sum_{m=0}^{n-2} C_m^{n-1}\,\partial^{-1}\Big[\partial^{-1}\Delta_+(U'_{(n-1-m)}\hat{\Lambda}_{(m)}\phi) - \partial^{-1}(U'_{(n-1-m)}\hat{\Lambda}_{(m)}\phi)U \\ + U'_{(n-1-m)}\,\partial^{-1}\Big(\Delta_+\hat{\Lambda}_{(m)}\phi - \hat{\Lambda}_{(m)}\phi U + \sum_{l=0}^{m} C_l^m U'_{(m-1)}\hat{\Lambda}_{(l)}\phi\Big)\Big] \\ n = 2, 3, \ldots\end{aligned} \tag{3.7}$$

where the operator $\hat{\Lambda}_{(1)}$ is given by (3.5). In virtue of (3.6) the relation (3.3) is

$$\hat{\chi}^{(F)}_{(n)} = \tfrac{1}{2}\partial^{-1}\left(\mathscr{L}^+\hat{\Lambda}_{(n)}\phi^{(F)}_{(0)} - \sum_{m=0}^{n} C_m^n U'_{(n-m)}\hat{\Lambda}_{(m)}\phi^{(F)}_{(0)}\right). \tag{3.8}$$

By analogous calculations one can show that

$$\check{\phi}^{(F)}_{(n)} = \check{\Lambda}_{(n)}\phi^{(F)}_{(0)} \qquad n = 1, 2, 3, \ldots \tag{3.9}$$

and

$$\check{\chi}^{(F)}_{(n)} = -\tfrac{1}{2}\partial^{-1}\Big(\mathscr{L}'\check{\Lambda}_{(n)}\phi^{(F)}_{(0)} - \sum_{m=0}^{n} C^n_m \check{\Lambda}_{(m)}\phi^{(F)}_{(0)} U_{(n-m)}\Big). \tag{3.10}$$

The recursion operators $\check{\Lambda}_{(n)}$ are found from the recursion relations

$$\begin{aligned}\check{\Lambda}_{(n)}\phi &= \check{\Lambda}_{(1)}\check{\Lambda}_{(n-1)}\phi \\ &\quad + \frac{1}{4\alpha}\sum_{m=0}^{n-2} C^{n-1}_m \partial^{-1}\Big[\partial^{-1}\Delta_-(\check{\Lambda}_{(m)}\phi U_{(n-1-m)}) - U'\partial^{-1}(\check{\Lambda}_{(m)}\phi U_{(n-1-m)}) \\ &\quad + \partial^{-1}\Big(\Delta_+\check{\Lambda}_{(m)}\phi - U'\check{\Lambda}_{(m)}\phi + \sum_{l=0}^{m} C^m_l \check{\Lambda}_{(l)}\phi \;\; U_{(m-l)}\Big)U_{(n-m)}\Big] \\ &\quad n = 2, 3, \ldots\end{aligned} \tag{3.11}$$

where $\check{\Lambda}_{(0)} \equiv 1$ and

$$\check{\Lambda}_{(1)}\phi = \frac{1}{4\alpha}\partial^{-1}[\partial^{-1}(\Delta_+\mathscr{L}'\phi - \Delta_-(\phi U)) + U'\partial^{-1}(\mathscr{L}'\phi - \phi U) + \partial^{-1}(\mathscr{L}^+\phi - U'\phi) \;\; U]. \tag{3.12}$$

The operators $\hat{\Lambda}_{(n)}$ and $\check{\Lambda}_{(n)}$ are not independent. From their definitions it follows that for example

$$\check{\Lambda}_{(n)} = \sum_{k=0}^{n} (-1)^k C^n_k \frac{\partial^{n-k}}{\partial y^{n-k}}\hat{\Lambda}_{(k)} \qquad n = 1, 2, \ldots. \tag{3.13}$$

In the following constructions we shall use the operators $\hat{\Lambda}^+_{(n)}$ and $\check{\Lambda}^+_{(n)}$ adjoint to the operators $\hat{\Lambda}_{(n)}$ and $\check{\Lambda}_{(n)}$ with respect to bilinear forms

$$\langle \chi, \psi \rangle \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} dx\, dy\, \mathrm{Tr}(\chi(x, y)\psi(x, y)).$$

The corresponding recursion relations for the operators $\hat{\Lambda}^+_{(n)}$ are of the form

$$\begin{aligned}\hat{\Lambda}^+_{(n)}\phi &= \hat{\Lambda}^+_{(n-1)}\hat{\Lambda}^+_{(1)}\phi - \frac{1}{4\alpha}\sum_{m=0}^{n-2} C^{n-1}_m \hat{\Lambda}^+_{(m)}\{\partial^{-1}[(\Delta_+ - U)\partial^{-1}\phi]U'_{(n-1-m)} \\ &\quad + (\Delta_- - U)\partial^{-1}(\partial^{-1}\phi U'_{(n-1-m)})\} \\ &\quad - \frac{1}{4\alpha}\sum_{m=0}^{n-2} C^{n-1}_m \sum_{l=0}^{m} C^m_l \hat{\Lambda}^+_l(\partial^{-1}(\partial^{-1}\phi U'_{(n-1-m)}) \;\; U'_{(m-l)}) \\ &\quad n = 2, 3, \ldots\end{aligned} \tag{3.14}$$

where the operator $\hat{\Lambda}^+_{(1)}$ acts as follows

$$\begin{aligned}\hat{\Lambda}^+_{(1)}\phi &= -\frac{1}{4\alpha}[\partial^{-1}\Delta^2_+\partial^{-1}\phi + U\Delta_+\partial^{-2}\phi + (\Delta_-\partial^{-2}\phi) \;\; U' + \Delta_+\partial^{-1}(U \;\; \partial^{-1}\phi) \\ &\quad + \Delta_-\partial^{-1}(\partial^{-1}\phi \;\; U') - U\partial^{-1}(\partial^{-1}\phi \;\; U' - U \;\; \partial^{-1}\phi) \\ &\quad + \partial^{-1}(\partial^{-1}\phi \;\; U' - U\partial^{-1}\phi) \;\; U'].\end{aligned} \tag{3.15}$$

In formulae (3.14), (3.15) and below $(\partial^{-1}f)(x, y) = \int_{-\infty}^{x} dx' f(x', y)$.

The operators $\check{\Lambda}^{+}_{(n)}$ can be found from the recursion relations analogous to (3.14) or from the relations

$$\check{\Lambda}^{+}_{(n)}=(-1)^{n}\sum_{k=0}^{n}C^{n}_{k}\hat{\Lambda}^{+}_{(k)}\frac{\partial^{n-k}}{\partial y^{n-k}}\qquad n=1,2,\ldots \tag{3.16}$$

4. General structure of the integrable equations and Bäcklund transformations

The existence of the recursion operators $\hat{\Lambda}_{(n)}$ and $\check{\Lambda}_{(n)}$ is extremely important for the generalisation of the AKNS method to the two-dimensional problem (3.1). With the use of the relations (3.3), (3.6), (3.8), (3.9), from (2.12) we obtain

$$\begin{aligned}\int_{-\infty}^{+\infty}\mathrm{d}x\,\mathrm{d}y\,\mathrm{Tr}\Big\{\phi^{(F)}_{(0)}(x,y,t)\sum_{\gamma=1}^{N^2}\sum_{n=0}^{\infty}\Big[&b_{1\gamma(n)}(t)\hat{\Lambda}^{+}_{(n)}UH_{\gamma}\\&-\tfrac{1}{2}b_{2\gamma(n)}(t)\Big(\hat{\Lambda}^{+}_{(n)}\mathscr{L}\partial^{-1}UH_{\gamma}-\sum_{m=0}^{n}C^{n}_{m}\hat{\Lambda}^{+}_{(m)}(\partial^{-1}UH_{\gamma}U'_{(n-m)})\Big)\\&-b_{1\gamma(n)}(t)(-1)^{n}\check{\Lambda}^{+}_{(n)}H_{\gamma}U'-\tfrac{1}{2}b_{2\gamma(n)}(t)(-1)^{n}\Big(\check{\Lambda}^{+}_{(n)}\mathscr{L}^{+\prime}H_{\gamma}\partial^{-1}U'\\&-\sum_{m=0}^{n}C^{n}_{m}\check{\Lambda}^{+}_{(m)}(U_{(n-m)}H_{\gamma}\partial^{-1}U')\Big)\Big]\Big\}=0\end{aligned}\tag{4.1}$$

where operators $\hat{\Lambda}^{+}_{(n)}$ and $\check{\Lambda}^{+}_{(n)}$ are given by formulae (3.14)–(3.16).

The equality (4.1) is fulfilled if

$$\begin{aligned}\sum_{\gamma=1}^{N^2}\Big(&(B_{1\gamma}(\hat{\Lambda}^{+},t)-\tfrac{1}{2}B_{2\gamma}(\hat{\Lambda}^{+},t)\mathscr{L}\partial^{-1})UH_{\gamma}-(B_{1\gamma}(\check{\Lambda}^{+},t)+\tfrac{1}{2}B_{2\gamma}(\check{\Lambda}^{+},t)\mathscr{L}^{+\prime}\partial^{-1})H_{\gamma}U'\\&+\tfrac{1}{2}\sum_{n=0}^{\infty}b_{2\gamma(n)}(t)\sum_{m=0}^{n}C^{n}_{m}(\hat{\Lambda}^{+}_{(m)}(\partial^{-1}UH_{\gamma}U'_{(n-m)})\\&+(-1)^{n}\check{\Lambda}^{+}_{(m)}(U_{(n-m)}H_{\gamma}\partial^{-1}U))\Big)=0\end{aligned}\tag{4.2}$$

where

$$B_{\frac{1}{2}\gamma}(\hat{\Lambda}^{+},t)\overset{\text{def}}{=}\sum_{n=0}^{\infty}b_{\frac{1}{2}\gamma(n)}(t)\hat{\Lambda}^{+}_{(n)},\qquad B_{\frac{1}{2}\gamma}(\check{\Lambda}^{+},t)\overset{\text{def}}{=}\sum_{n=0}^{\infty}b_{\frac{1}{2}\gamma(n)}(t)(-1)^{n}\check{\Lambda}^{+}_{(n)}.$$

If the quantities $\phi^{(F)}_{(0)}(x,y,t)$ form a complete set (similar to the one-dimensional case $\partial U/\partial y=0$) then the equality (4.2) is also a necessary condition of fulfilment of the equality (4.1).

The relation (4.2) just determines the transformation of the potential $U(x,y,t)\to U'(x,y,t)$ which corresponds to the transformation of the scattering matrix $\hat{S}(\tilde{\lambda},\lambda,t)\to\hat{S}'(\tilde{\lambda},\lambda,t)$ of the form (2.9). It is important that the relation (4.2) contains only the potential U and transformed potential U'.

The transformations (4.2) form, as it is easy to see from (2.9), an infinite-dimensional group. This group of transformations which acts on the manifold of the potentials $\{U(x,y,t)\}$ by the formula (4.2) and on the manifold of the scattering matrices $\{\hat{S}(\tilde{\lambda},\lambda,t)\}$ by the formula (2.9) plays a fundamental role in the analysis of the nonlinear systems connected with the problem (2.1) (or (1.1)) and their properties.

Let us consider the transformation (2.9), (4.2) generated by the infinitesimal displacement in time $t: t \to t' = t+\varepsilon$, $\varepsilon \to 0$. For this transformation

$$U'(x, y, t) = U(x, y, t') = U(x, y, t) + \varepsilon \frac{\partial U(x, y, t)}{\partial t'}$$
$$B_1 = C_1 \equiv I_N, \qquad B_{2\gamma}(\lambda^2, t) = C_{2\gamma}(\lambda^2, t) = -\varepsilon\,\Omega_\gamma(\lambda^2, t) \qquad \gamma = 1, \ldots, N^2 \tag{4.3}$$

where $\Omega_\gamma(\lambda^2, t) = \Sigma_{n=0}^{\infty}\,\omega_{\gamma n}(t)\lambda^{2n}$ and $\omega_{\gamma n}(t)$ are arbitrary functions. Substituting (4.3) into (4.2) and keeping the terms of first order in ε we obtain an evolution equation

$$\frac{\partial U(x, y, t)}{\partial t} - \tfrac{1}{2}\sum_{\gamma=1}^{N^2}(\Omega_\gamma(\hat{L}^+, t)\mathscr{L}\partial^{-1}UH_\gamma + \Omega_\gamma(\check{L}^+, t)\mathscr{L}^+H_\gamma\partial^{-1}U)$$
$$+\tfrac{1}{2}\sum_{\gamma=1}^{N^2}\sum_{n=0}^{\infty}\omega_{\gamma n}(t)\sum_{m=0}^{n}C^n_m(\hat{L}^+_{(m)}(\partial^{-1}UH_\gamma U_{(n-m)})$$
$$+(-1)^n\check{L}^+_{(m)}(U_{(n-m)}H_\gamma\partial^{-1}U)) = 0 \tag{4.4}$$

where

$$\hat{L}^+_{(n)} \stackrel{\text{def}}{=} \hat{\Lambda}^+_{(n)}(U' = U), \qquad \check{L}^+_{(n)} \stackrel{\text{def}}{=} \check{\Lambda}^+_{(n)}(U' = U)$$

$$\Omega_\gamma(\hat{L}^+, t) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty}\omega_{\gamma n}(t)\hat{L}^+_{(n)},$$

$$\Omega_\gamma(\check{L}^+, t) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty}\omega_{\gamma n}(t)(-1)^n\check{L}^+_{(n)}.$$

The operators $\hat{L}^+_{(n)}$ and $\check{L}^+_{(n)}$ are calculated from the recursion relations (3.14) and from (3.16) at $U' = U$. For example

$$\hat{L}^+_{(1)}\cdot = -\frac{1}{4\alpha}\left\{\left(\alpha\frac{\partial}{\partial y}\partial^{-1} + \frac{\partial}{\partial x}\right)^2 + 2[U(x, y), \cdot]_+\right.$$
$$+\left[\frac{\partial U(x, y)}{\partial x}, \partial^{-1}\cdot\right]_+ - \alpha\left[U(x, y), \frac{\partial}{\partial y}\partial^{-2}\cdot\right]_-$$
$$\left.+\alpha\frac{\partial}{\partial y}\partial^{-1}[U, \partial^{-1}\cdot]_- + [U(x, y), \partial^{-1}[U, \partial^{-1}\cdot]_-]_-\right\} \tag{4.5}$$

and $\check{L}^+_{(1)} = -\hat{L}^+_{(1)} - \partial/\partial y$ where $[A, B]_\pm \stackrel{\text{def}}{=} AB \pm BA$.

For the scattering matrix from (2.9) we correspondingly obtain the following linear evolution equation

$$\frac{d\hat{S}(\tilde{\lambda}, \lambda, t)}{dt} = Y(\tilde{\lambda}, t)\hat{S}(\tilde{\lambda}, \lambda, t) - \hat{S}(\tilde{\lambda}, \lambda, t)Y(\lambda, t) \tag{4.6}$$

where

$$Y(\lambda, t) = -i\sqrt{\alpha}\lambda\begin{pmatrix}\sum_{\gamma=1}^{N^2}\Omega_\gamma(\lambda^2, t)H_\gamma & 0 \\ 0 & -\sum_{\gamma=1}^{N^2}\Omega_\gamma(\lambda^2, t)H_\gamma\end{pmatrix}. \tag{4.7}$$

Thus the nonlinear evolution equations (4.4) are infinitesimal forms of the transformations (4.2) generated by time displacement. The class of equations (4.4) is characterised by the integer N, recursion operators $\hat{L}^{+}_{(n)}$, $\check{L}^{+}_{(n)}$ and by N^2 arbitrary functions $\Omega_1(\lambda^2, t), \ldots, \Omega_{N^2}(\lambda^2, t)$ entire on λ^2.

The equations (4.4) are just the nonlinear evolution equations in $1+2$ dimensions (one time and two spatial) integrable by the IST method with help of the linear problem (1.1) (or (2.1)). Using the two-dimensional version of the IST method (see e.g. Zakharov *et al* 1980, Zakharov and Shabat 1974, Zakharov and Manakov 1978, Zakharov 1980) one can, in principle, find a broad class of exact solutions of equation (4.4). Let us note that the evolution law of the scattering matrix of the type (4.7) was earlier considered in Zakharov and Manakov 1978, Zakharov 1980.

The simplest equation of the form (4.4) which corresponds to

$$\sum_{\gamma=1}^{N^2} H_\gamma \omega_{\gamma 0}(t) = \omega_0(t) I_N, \qquad \sum_{\gamma=1}^{N^2} H_\gamma \omega_{\gamma 1}(t) = \omega_1(t) I_N$$

where $\omega_0(t)$ and $\omega_1(t)$ are scalar functions and $\omega_{\gamma 2} = \omega_{\gamma 3} = \ldots = 0$ ($\gamma = 1, \ldots, N^2$) is

$$\frac{\partial U(x,y,t)}{\partial t} - \omega_0(t)\frac{\partial U(x,y,t)}{\partial x} + \frac{\omega_1(t)}{4\alpha}\left(\frac{\partial^3 U}{\partial x^3} + 3\alpha^2 \int_{-\infty}^{x} \mathrm{d}x' \frac{\partial^2 U(x',y,t)}{\partial y^2} + 3\frac{\partial}{\partial x}(U^2(x,y,t)) + 3\left[U(x,y,t), \int_{-\infty}^{x} \mathrm{d}x' \frac{\partial U(x',y,t)}{\partial y}\right]_{-}\right) = 0. \tag{4.8}$$

In the scalar case ($N=1$) and constant ω_0, $\omega_1 = -4\alpha$ equation (4.8) is the well known Kadomtsev–Petviashvili (KP) equation considered in Zakharov and Shabat (1974), Dryuma (1974), Zakharov and Manakov (1979), Manakov (1981). At arbitrary N it is the matrix KP equation which was discussed (see e.g. Chudnovsky 1980). The KP equation (4.8) is the lowest (KP_1) form of the infinite family (KP family) of the $1+2$ dimensional equations (4.4) (KP_n: $\Sigma_{\gamma=1}^{N^2} H_\gamma \omega_{\gamma 0} = \omega_0 I_N$, $\Sigma_{\gamma=1}^{N^2} H_\gamma \omega_{\gamma n} = -\alpha 2^{2n} I_N$, $\omega_{\gamma 1} = \omega_{\gamma 2} = \ldots = \omega_{\gamma n-1} = \omega_{\gamma n+1} = \ldots = 0$, $n = 1, 2, 3, \ldots$).

In the one-dimensional case $\partial U/\partial y = 0$, the equations (4.4) coincide with those integrable by the matrix stationary Schrödinger spectral problem (Calogero and Degasperis 1977).

5. The Bäcklund transformation group and nonlinear superposition principle

The infinite-dimensional group of transformations (4.2) contains all transformations specific to the integrable equations (4.4).

Let us consider transformations (4.2) with matrices B and C commuting with the matrix $Y(\lambda, t)$, (4.7). At $\partial B_1/\partial t = \partial B_2/\partial t = \partial C_1/\partial t = \partial C_2/\partial t = 0$ these transformations do not change the evolution law (4.6) of the scattering matrix and, therefore, they are auto Bäcklund transformations (BT) for the equations (4.4): they transform solutions of the definite equations of the form (4.4) into the solutions of the same equation. If $\partial B_1/\partial t \neq 0$, $\partial B_2/\partial t \neq 0$ then the transformations (4.2) are generalised BT. The infinite-dimensional group of transformations (4.2) also contains, as a subgroup, an infinite-dimensional symmetry group of the equations (4.4). Group theoretical structure of these equations will be considered elsewhere.

Let us consider the simplest BT (4.2) which corresponds to

$$\sum_{\gamma=1}^{N^2} H_\gamma b_{1\gamma(0)} = b_1 I_N, \qquad \sum_{\gamma=1}^{N^2} H_\gamma b_{2\gamma(0)} = b_2 I_N$$

and $b_{1\gamma(n)} = b_{2\gamma(n)} = 0$ $(n = 1, 2, \ldots)$ where b_1 and b_2 are arbitrary constants. This BT B_b is

$$\begin{aligned} B_b(U \to U'){:}\quad & b(U'-U) + \frac{\partial}{\partial x}(U'+U) - \alpha \int_{-\infty}^{x} \mathrm{d}x' \frac{\partial}{\partial y}(U'(x',y,t) - U(x',y,t)) \\ & + \int_{-\infty}^{x} \mathrm{d}x' \, (U'(x',y,t) - U(x',y,t) \cdot U'(x,y,t) \\ & - U(x,y,t) \int_{-\infty}^{x} \mathrm{d}x'(U'(x',y,t) - U(x',y,t)) = 0 \end{aligned} \tag{5.1}$$

where $b = 2b_1/b_2$. Introducing the potential $W(x, y, t)$ by $U(x, y, t) = \partial W(x, y, t)/\partial x$ $(W(-\infty, y, t) = 0)$ we obtain a local form of BT (5.1):

$$\begin{aligned} B_b(W \to W'){:}\quad & b\frac{\partial}{\partial x}(W'-W) + \frac{\partial^2}{\partial x^2}(W'+W) - \alpha \frac{\partial}{\partial y}(W'-W) \\ & + (W'-W)\frac{\partial W'}{\partial x} - \frac{\partial W}{\partial x}(W'-W) = 0. \end{aligned} \tag{5.2}$$

Let us note that BT (5.2) is universal i.e. it is a BT for any equation of the form (4.4) and in particular for any equation from the KP family.

BT (5.2) allows us to construct an infinite family of the solutions of the equations (4.4) by almost pure algebraic operations. Indeed let us consider the following diagram

$$\begin{array}{ccccc} & & W_1 & & \\ & \nearrow^{B_{b_1}} & & \searrow^{B_{b_2}} & \\ W_0 & & & & W_3 \\ & \searrow_{B_{b_2}} & & \nearrow_{B_{b_1}} & \\ & & W_2 & & \end{array} \tag{5.3}$$

which expresses the commutativity of BT (5.2) with different parameters $b{:} B_{b_1}B_{b_2} = B_{b_2}B_{b_1}$. Here $U_i = \partial W_i/\partial x$ $(i = 0, 1, 2, 3)$ are four solutions of the definite (but any) equation of the form (4.4). With the use of the equation (5.2) for all four solutions W_0, W_1, W_2, W_3; from (5.3) we obtain

$$\begin{aligned} W_3 = {} & (b_1 - b_2 + W_1 - W_2)^{-1}[(b_1 - b_2)(W_1 + W_2 - W_0) \\ & - W_0(W_1 - W_2) + 2\partial(W_1 - W_2)/\partial x + W_1^2 - W_2^2]. \end{aligned} \tag{5.4}$$

Therefore with the three solutions given, W_0, W_1, W_2, one can easily calculate the fourth solution W_3 from (5.4). Let us emphasise that the relation (5.4) is a universal one, i.e. it is valid for all the equations of the form (4.4) and in particular for any equation from the KP family.

The relation (5.4) is just the nonlinear superposition principle for the equations (4.4). Some concrete nonlinear superposition formulae for some concrete 1+1 dimensional equations are well known (see e.g. Miura 1976).

Starting from the trivial solution $W_0 = 0$ and using the simplest one-soliton solutions W_1 and W_2 (they differ only by the value of the constant b) one can easily obtain with the use of (5.4) the infinite family of the soliton-type solutions of the equations of the form (4.4).

In the scalar case ($N=1$) the BT (5.2) and nonlinear superposition formula (5.4) reduce to

$$b(W'-W)+\frac{\partial}{\partial x}(W'+W)-\alpha\int_{-\infty}^{x}\mathrm{d}x'\,\frac{\partial}{\partial y}(W'(x',y)-W(x',y))+\tfrac{1}{2}(W'-W)^2=0 \tag{5.5}$$

and

$$W_3=W_1+W_2-W_0+2\frac{\partial}{\partial x}\ln(b_1-b_2+W_1-W_2) \tag{5.6}$$

which coincide at $b=0$ with those found earlier by another method in Chen (1975). In Chen (1975) the solutions W_1, W_2 were calculated with the use of BT (5.5) (at $N=1, b=0$).

In the scalar case ($N=1$) one can also obtain from (5.3) the other nonlinear superposition formula for BT (5.5). It is

$$W_3=W_0+\frac{(b_1+b_2)(W_2-W_1)-2\alpha\int_{-\infty}^{x}\mathrm{d}x'\,(\partial/\partial y)(W_2(x',y)-W_1(x',y))}{b_1-b_2+W_1-W_2}$$

which at $\partial W/\partial y=0$ reduces to the well known superposition formula for the KDV family of equations (see e.g. Miura 1976).

In the one-dimensional case $\partial U/\partial y=0$ the general BT (4.2) coincide with those connected with the stationary matrix Schrödinger spectral problem (Calogero and Degasperis 1977).

6. Conclusion

In the conclusion we emphasise the following points.

(i) All the results of the present paper can be generalised to the case when $\lim_{\sqrt{x^2+y^2}\to\infty} U(x,y,t)\neq 0$. In particular, the equation (4.8), BT (5.2) and superposition formula (5.4) remain unchanged.

(ii) Let us emphasise that in the present paper we consider another direct scattering problem for the spectral problem (1.1) (or (2.1)) than in Zakharov and Manakov (1979), Manakov (1981). Namely, in our approach the scattering matrix S, in essence, relates the asymptotics of the solutions ψ of the problem (1.1) on x infinities, i.e. at $x=-\infty$ and $x=+\infty$, while in Zakharov and Manakov (1979) and Manakov (1981) the standard version of the scattering problem for the non-stationary Schrödinger equation (1.1) is used in which the scattering matrix connects the solutions on y infinities, i.e. at $y=-\infty$ and $y=+\infty$. The interrelation between these two approaches will be considered elsewhere.

(iii) One can try to transfer the $1+2$ dimensional AKNS technique to other two-dimensional spectral problems. In particular, to

$$\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}+U(x,y,t)\psi=0 \tag{6.1}$$

and

$$\frac{\partial\psi}{\partial x}+\begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix}\left(\alpha\frac{\partial^2}{\partial y^2}+2\beta\frac{\partial}{\partial y}\right)\psi+\begin{pmatrix} 0 & Q(x,y,t) \\ R(x,y,t) & 0 \end{pmatrix}\left(\alpha\frac{\partial}{\partial y}+\beta\right)\psi=0 \tag{6.2}$$

where U, Q and R are $N\times N$ matrices and α and β are arbitrary constants. For the one-dimensional counterparts of the problems (6.1) and (6.2) (i.e. at $\partial U/\partial y=0$, $\partial Q/\partial y=\partial R/\partial y=0$ and $\partial\psi/\partial y=\lambda\psi$) it is possible to calculate the recursion operators, to find the general form of the integrable equations and their Backlund transformations and so on (for problem (6.1) see Calogero and Degasperis (1977), for problem (6.2) see Konopelchenko (1981c)). For the two-dimensional problems (6.1) and (6.2) one can obtain all the formulae analogous to those given in § 2. But the recursion operators analogous to $\hat{\Lambda}_{(n)}$ and $\check{\Lambda}_{(n)}$ do not exist for the problems (6.1) and (6.2). Namely, instead of the relations (3.6) and (3.9) we obtain the relations of the type

$$\hat{\phi}^{(F)}_{(n)}=\hat{\Lambda}_{0(n)}\phi^{(F)}_{(0)}+\hat{\Lambda}_{1(n)}\hat{\phi}^{(F)}_{(1)}$$

$$\check{\phi}^{(F)}_{(n)}=\check{\Lambda}_{0(n)}\phi^{(F)}_{(0)}+\check{\Lambda}_{1(n)}\hat{\phi}^{(F)}_{(1)} \qquad n=2,3,4,\ldots$$

where $\hat{\Lambda}_{0(n)}$, $\hat{\Lambda}_{1(n)}$ and $\check{\Lambda}_{0(m)}$, $\check{\Lambda}_{1(n)}$ are operators which can be calculated by certain recursion relations. So, the two-dimensional spectral problems (6.1) and (6.2) essentially differ from the problem (1.1) and the problem $\partial\psi/\partial x+A\partial\psi/\partial y+P(x,y,t)\psi=0$ (Konopelchenko 1981d) from the AKNS-technique point of view.

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